

# Mechanics: Gravitation (Solutions)

FIZIKA SPhO Training

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# 1 Notes

The gravitational force is one of the four **fundamental forces** in nature. It is a mutual interaction between all massive particles, and it is always attractive in nature.

## 1.1 The Basic Quantities

In gravitation, the four basic quantities you will deal with are:

1. Gravitational Force,  $\mathbf{F}_g$
2. Gravitational Field,  $\mathbf{g}$
3. Gravitational Potential Energy,  $U_g$
4. Gravitational Potential,  $\phi_g$

Note that the first two are vectors while the other two are scalars.

### 1.1.1 Gravitational Force and Gravitational Field

Consider two point masses  $M_1$  and  $M_2$  located at position vectors  $\mathbf{r}_1$  and  $\mathbf{r}_2$ . Let the separation vector be  $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ .

The gravitational forces by each point mass on the other mass are given by

$$\mathbf{F}_{g,1\text{ by }2} = \frac{GM_1M_2}{|\mathbf{r}|^3}\mathbf{r} \quad (1)$$

$$\mathbf{F}_{g,2\text{ by }1} = -\frac{GM_1M_2}{|\mathbf{r}|^3}\mathbf{r} \quad (2)$$

The gravitational field can be thought of as the gravitational force per unit mass. Let the mass  $M$  be placed at the origin. The gravitational field at a position vector  $\mathbf{r}$  away is given by

$$\mathbf{g} = -\frac{GM}{|\mathbf{r}|^3}\mathbf{r} \quad (3)$$

### 1.1.2 Gravitational Potential Energy and Gravitational Potential

Consider two point masses again. The gravitational potential energy is given by

$$U_g = -\frac{GM_1M_2}{|\mathbf{r}|} \quad (4)$$

The gravitational potential can be thought of as the gravitational potential energy per unit mass. Thus, the gravitational potential at a position vector  $\mathbf{r}$  away is given by

$$\phi_g = -\frac{GM}{|\mathbf{r}|} \quad (5)$$

The **negative signs** in Equations (4) and (5) are very important! The signs are negative as the gravitational force is always an attractive interaction.

### 1.1.3 Relationships Between Quantities

As we have studied before, force is the negative of the derivative of potential energy. Thus,

$$F_g = -\frac{dU_g}{dr} \quad (6)$$

Conversely, to get gravitational potential energy from gravitational force,

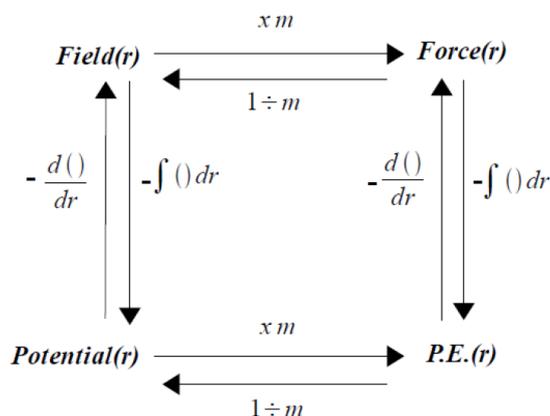
$$U_g = -\int_{\infty}^{\mathbf{r}} \mathbf{F}_g \cdot d\mathbf{r} \quad (7)$$

Since the gravitational field and gravitational potential can be thought of as the gravitational force and gravitational potential energy per unit mass respectively, we also have

$$g = -\frac{d\phi_g}{dr} \quad (8)$$

$$\phi_g = -\int_{\infty}^{\mathbf{r}} \mathbf{g} \cdot d\mathbf{r} \quad (9)$$

A helpful image for remembering this is the square below:



## 1.2 Circular Orbits

You should already be familiar with this from H2 physics - this is just a recap.

### 1.2.1 Period, Speed and Radius of Circular Orbits

Consider a satellite of mass  $m$  in a **circular orbit** around a planet of mass  $M$ . We shall assume  $M \gg m$  (which is justifiable for a normal satellite and normal planet), so that the planet is approximately stationary at the centre of the orbit.

Since the gravitational force provides for the centripetal force,

$$F_c = F_g \Rightarrow \frac{mv^2}{R} = \frac{GMm}{R^2} \Rightarrow v = \sqrt{\frac{GM}{R}} \quad (10)$$

Thus, from circular motion,

$$\omega = \frac{v}{R} = \sqrt{\frac{GM}{R^3}}, \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^3}{GM}} \quad (11)$$

Remember that none of these quantities depend on the mass of the satellite,  $m$ !

We are mostly interested in **geostationary orbits**. This is a special type of orbit centered around Earth, where the satellite remains **fixed** above a specific point on the Earth's equator.

The period of the geostationary orbit is equal to the period of the Earth's rotation:

$$T_{\text{geostationary}} = T_{\text{Earth}} \approx 24 \text{ h} \quad (12)$$

You can show that the radius of geostationary orbit is around  $R_{\text{geostationary}} = 42000 \text{ km}$ .

### 1.2.2 Energy of Circular Orbits

The total energy of a circular orbit consists of kinetic energy and gravitational potential energy. As there are no external forces acting on the satellite-planet system, **total energy is conserved**.

We may calculate the different forms of energy as such:

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2R}, \quad U_g = -\frac{GMm}{R}, \quad E = K + U_g = -\frac{GMm}{2R} = \text{constant} \quad (13)$$

Note that the total energy  $E < 0$ . This implies that the orbit is **bounded**. We'll see more on what bounded orbits mean in Section 1.6.

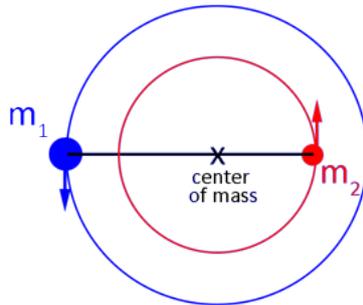
**Only for circular orbits**, we may write:

$$K = -E, \quad U_g = 2E, \quad U_g = -2K \quad (14)$$

These relations may be useful if you need to find one form of energy given another.

### 1.2.3 Binary Star Systems

If the masses of the two gravitationally interacting objects are comparable (i.e.  $M$  is not  $\gg m$ ), then both objects will revolve around each other, **about their CM**. The orbits are concentric circles, as shown in the diagram below.



Firstly, note that the angular frequency of both orbits must be the same, so that the CM stays at the same position. (The CM cannot move as there are no net external forces on the system.)

$$\omega_1 = \omega_2 = \omega \quad (15)$$

Let the radius of orbit of  $m_1$  be  $r_1$ , and that of  $m_2$  be  $r_2$ , and  $r = r_1 + r_2$ . Then,

$$m_1 r_1 \omega_1^2 = \frac{Gm_1 m_2}{r^2}, \quad m_2 r_2 \omega_2^2 = \frac{Gm_1 m_2}{r^2} \quad (16)$$

From the CM relation, we have

$$\frac{r_2}{r_1} = \frac{m_1}{m_2} \quad (17)$$

Using Equations (15), (16) and (17), we can obtain that

$$\omega = \sqrt{\frac{G(m_1 + m_2)}{r^3}} \quad (18)$$

We can analyse this using **reduced mass** as well. Recall that the system is equivalent to a mass  $\mu = \frac{m_1 m_2}{m_1 + m_2}$  interacting with a mass  $M = m_1 + m_2$  at rest. You can then recover the same results. (Try it yourself as an exercise!)

### 1.3 Elliptical Orbits

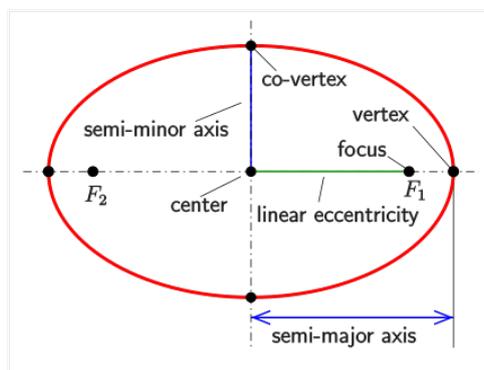
In reality, most orbits are not circular. They are instead **elliptical!**

#### 1.3.1 Ellipse Geometry

An ellipse centred at the origin can be described by the Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (19)$$

where  $a$  and  $b$  ( $a > b$ ) are the semi-major and semi-minor axes of the ellipse respectively.



The **foci** (labelled  $F_1$  and  $F_2$ ) of the ellipse are located at  $(\pm c, 0)$ , where

$$c = \sqrt{a^2 - b^2} \quad (20)$$

The value of  $c$  is also known as the **linear eccentricity**.

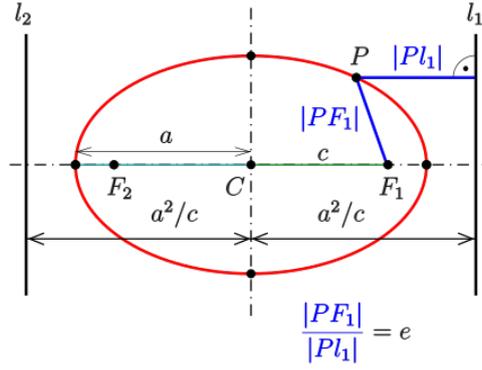
The foci are special points such that for all points  $P$  on the ellipse, they satisfy

$$PF_1 + PF_2 = 2a = \text{constant} \quad (21)$$

The **eccentricity**  $\varepsilon$  (different from linear eccentricity!) of the ellipse is defined as

$$\varepsilon = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}} \quad (22)$$

The **directrix** of the ellipse is a special line lying *outside* the ellipse.



As per the diagram above, the directrix is located at

$$x_{directrix} = \frac{a^2}{c} \tag{23}$$

and it satisfies

$$\frac{|PF_1|}{|Pl_1|} = \varepsilon \tag{24}$$

for all points  $P$  on the ellipse.

You can visualise for yourself that that a circle has  $\varepsilon \rightarrow 0^+$  and a line has  $\varepsilon \rightarrow 1^-$ .

### 1.3.2 Polar Equation of Elliptical Orbits

Consider a satellite of mass  $m$  in an **elliptical orbit** around a planet of mass  $M$ . Again, we assume that  $M \gg m$  so that the planet is approximately stationary.

The polar equation of the ellipse, with the origin at the position of the planet, is

$$r = \frac{p}{1 - \varepsilon \cos \theta}, \quad 0 < \varepsilon < 1, \quad p > 0 \tag{25}$$

We may transform Equation (25) into Cartesian coordinates. Recall that  $r^2 = x^2 + y^2$ , and  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Hence,

$$r = p + \varepsilon r \cos \theta \quad \Rightarrow \quad r^2 = x^2 + y^2 = (p + \varepsilon x)^2 \quad \Rightarrow \quad \frac{\left(x - \frac{p\varepsilon}{1-\varepsilon^2}\right)^2}{\left(\frac{p}{1-\varepsilon^2}\right)^2} + \frac{y^2}{\left(\frac{p^2}{1-\varepsilon^2}\right)} = 1 \tag{26}$$

Recall from H2 math that this is an ellipse with origin shifted to the right by  $\frac{p\varepsilon}{1-\varepsilon^2}$  units along the positive  $x$ -axis. Comparing to the form of the ellipse equation in Equation (19), we have

$$a = \frac{p}{1 - \varepsilon^2}, \quad b = \frac{p}{\sqrt{1 - \varepsilon^2}} \tag{27}$$

Thus, we may define this constant  $p$  from Equation (25) as

$$p = \frac{b^2}{a} = \frac{a^2 - c^2}{a} \tag{28}$$

### 1.3.3 Apsides of Elliptical Orbits

Based on Equation (25), we can see that by varying  $\theta$ ,

$$r_{max} = \frac{p}{1 - \varepsilon}, \quad r_{min} = \frac{p}{1 + \varepsilon} \quad (29)$$

The *existence* of a maximum and minimum radial distance means that the orbit is **bounded**.

The maximum and minimum distances occur at the points named the **apsides**. There are two types of apsides - the **periapsis** (nearest point) and the **apoapsis** (furthest point).

**Remark.** In some books, you may see the words "perihelion" and "aphelion", or "perigee" and "apogee". These are equivalent to the periapsis and apoapsis for the specific cases of a body orbiting the Sun and the Earth respectively.

### 1.3.4 Angular Momentum of Elliptical Orbits

Recall from the rotation handout that the angular momentum is defined as

$$\mathbf{L} = m\mathbf{r} \times \mathbf{v} \quad (30)$$

From Equations (1) and (2), the gravitational force acts purely in the radial direction (with the planet/Sun that is being revolved around as the origin). Thus, it contributes to *no net torque* around the central axis of the planet/Sun. Hence, **angular momentum is conserved!**

For an elliptical orbit, we are most interested in the angular momentum at the apsides, since the cross product in Equation (30) is easiest to calculate as  $\mathbf{r} \perp \mathbf{v}$ .

Thus, by COAM, letting subscripts  $p$  and  $a$  denote the periapsis and apoapsis respectively,

$$mr_p v_p = mr_a v_a \quad \Rightarrow \quad r_p v_p = r_a v_a \quad (31)$$

### 1.3.5 Energy of Elliptical Orbits

Similar to a circular orbit, the total energy of an elliptical orbit consists of kinetic energy and gravitational potential energy. Again, as there are no external forces acting on the system, total energy is conserved. However, the **radial distance isn't constant!**

The total energy for an elliptical orbit of semi-major axis  $a$  is instead given by

$$E = K + U_g = -\frac{GMm}{2a} = \text{constant} \quad (32)$$

The proof of Equation (32) is in the Appendix. However, it suffices to just quote it.

In many elliptical orbit questions, you will need to use Equation (31) and (32) in tandem. The example below illustrates how.

**Example 1.1** (Ricardo). A satellite of mass  $m_s$  is in an elliptical orbit around a planet of mass  $m_p$  ( $m_p \gg m_s$ ), which is located at one focus of the ellipse. The satellite has a speed  $v_a$  at the distance  $r_a$  when it is furthest from the planet. The distance of closest approach is  $r_p$ . Find the ratio  $\frac{r_a}{r_p}$  in terms of  $G, m_p, v_a$  and  $r_a$ .

Applying COAM at the periapsis and apoapsis,

$$r_p v_p = r_a v_a \quad \Rightarrow \quad v_p = \frac{r_a v_a}{r_p}$$

Applying COE at the periapsis and apoapsis,

$$\frac{1}{2}mv_p^2 - \frac{Gm_p m}{r_p} = \frac{1}{2}mv_a^2 - \frac{Gm_p m}{r_a} \Rightarrow v_p^2 - v_a^2 = 2Gm_p \left( \frac{1}{r_p} - \frac{1}{r_a} \right)$$

To eliminate  $v_p$ , substitute the first equation into the second:

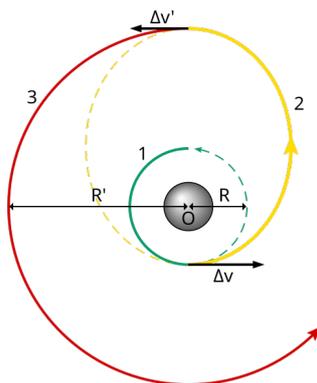
$$\begin{aligned} \left( \frac{r_a^2}{r_p^2} - 1 \right) v_a^2 &= 2Gm_p \left( \frac{r_a - r_p}{r_p r_a} \right) \Rightarrow \left( \frac{r_a^2 - r_p^2}{r_p^2} \right) v_a^2 = 2Gm_p \left( \frac{r_a - r_p}{r_p r_a} \right) \\ \Rightarrow \left( \frac{(r_a - r_p)(r_a + r_p)}{r_p^2} \right) v_a^2 &= 2Gm_p \left( \frac{r_a - r_p}{r_p r_a} \right) \Rightarrow \left( \frac{r_a + r_p}{r_p} \right) v_a^2 = \frac{2Gm_p}{r_a} \end{aligned}$$

Let our desired ratio  $\frac{r_a}{r_p} = x$ . Thus,

$$(x + 1) v_a^2 = \frac{2Gm_p}{r_a} \Rightarrow x = \frac{2Gm_p}{r_a v_a^2} - 1$$

### 1.3.6 Hohmann Transfer Orbit

The Hohmann transfer Orbit is a type of elliptical orbit, used as an intermediate to transfer a satellite from one circular orbit to another.



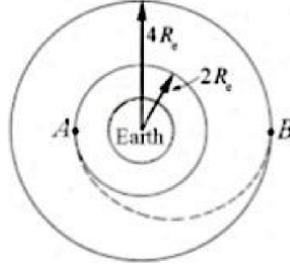
The Hohmann transfer orbit is labelled "2" in the diagram above. (It actually only consists of *half* the elliptical orbit.)

The key features of the Hohmann transfer orbit are:

1. It consists of **two speed changes**,  $\Delta v > 0$  and  $\Delta v' < 0$ , at the start and end respectively. These are usually achieved by rocket propulsion.
2. The semi-major axis of the Hohmann transfer orbit is given by  $a = \frac{R+R'}{2}$ .
3. The satellite starts at the periapsis and ends at the apoapsis.

**Remark.** In SPhO, you are **expected** to know what this refers to. That is, you might *not* be given a diagram for Hohmann transfer orbit questions!

**Example 1.2** (Ricardo). A space vehicle is in a circular orbit around the Earth of radius  $R_e$  and mass  $M_e$ . The mass of the vehicle is  $m_s \ll M_e$  and the radius of the orbit is  $2R_e$ . We want to transfer the vehicle to a circular orbit of radius  $4R_e$ . (i) Find the minimum energy expenditure for the Hohmann transfer. (ii) What changes in speed are required at points A and B in the Hohmann transfer (refer to the diagram below)?



(i) We may just consider the *change* in total energy of the two orbits:

$$\Delta E_{min} = E_f - E_i = -\frac{GM_em_s}{2(4R_e)} - \left(-\frac{GM_em_s}{2(2R_e)}\right) = \frac{GM_em_s}{8R_e}$$

The positive sign makes sense as we need to *supply* energy for the transfer.

(ii) The semi-major axis of the Hohmann transfer orbit is

$$a = \frac{2R_e + 4R_e}{2} = 3R_e$$

The total energy of an elliptical orbit with this semi-major axis is

$$E_{new} = -\frac{GM_em_s}{2a} = -\frac{GM_em_s}{2(3R_e)} = -\frac{GM_em_s}{6R_e}$$

The total energy is conserved, so we may calculate the KE at points A and B using the total energy and the GPE.

At the periapsis (point A),

$$-\frac{GM_em_s}{6R_e} = -\frac{GM_em_s}{2R_e} + \frac{1}{2}m_s v_A^2 \Rightarrow v_A = \sqrt{\frac{2GM_e}{3R_e}}$$

and at the apoapsis (point B),

$$-\frac{GM_em_s}{6R_e} = -\frac{GM_em_s}{4R_e} + \frac{1}{2}m_s v_B^2 \Rightarrow v_B = \sqrt{\frac{GM_e}{6R_e}}$$

The speeds in the circular orbits are  $v_{A,circular} = \sqrt{\frac{GM_e}{2R_e}}$  and  $v_{B,circular} = \sqrt{\frac{GM_e}{4R_e}}$ . Thus, the respective changes in speed are

$$\Delta v_A = \sqrt{\frac{2GM_e}{3R_e}} - \sqrt{\frac{GM_e}{2R_e}} = \left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}\right) \sqrt{\frac{GM_e}{R_e}}$$

$$\Delta v_B = \sqrt{\frac{GM_e}{6R_e}} - \sqrt{\frac{GM_e}{4R_e}} = \left(\sqrt{\frac{1}{6}} - \sqrt{\frac{1}{4}}\right) \sqrt{\frac{GM_e}{R_e}}$$

As expected,  $\Delta v_A > 0$  and  $\Delta v_B < 0$ .

## 1.4 Kepler's Laws

Kepler's three laws are essential to our understanding of gravitation and planetary motion. The detailed proofs of each law can be found in the Appendix.

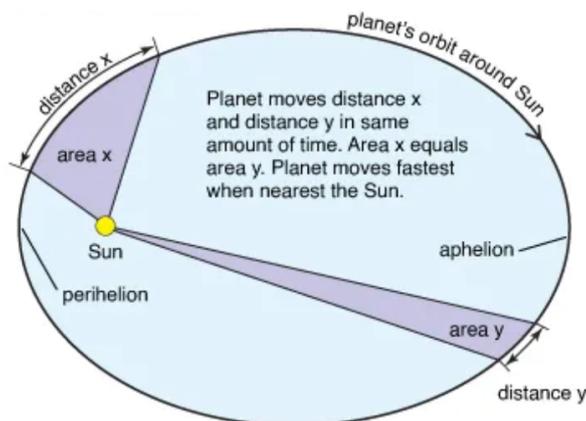
### 1.4.1 Kepler's First Law

**Statement:** The orbit of every planet is an **ellipse** with the Sun at one of the two foci.

This is self-explanatory. The other focus of the ellipse is what we call the **empty focus**.

### 1.4.2 Kepler's Second Law

**Statement:** A line joining the planet to the Sun sweeps equal areas in equal times.



This means that  $\frac{dA}{dt} = \text{constant}$ , or alternatively, by integrating over one period, we may write

$$\pi ab = \frac{LT}{2m} \quad (33)$$

where  $L$  is the angular momentum about the axis of the Sun,  $T$  is the period, and  $m$  is the mass of the planet.

This is useful for problems where the area swept out by an orbit can be easily calculated.

### 1.4.3 Kepler's Third Law

**Statement:** The square of the orbital period is proportional to the cube of the semi-major axis.

Mathematically, we may write

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}} \quad (34)$$

where  $T$  is the period,  $a$  is the semi-major axis, and  $M$  is the mass of the Sun.

Clearly, Kepler's Third Law only applies for **bounded orbits**, as we can only define a period for a bounded orbit and not an orbit that goes to infinity.

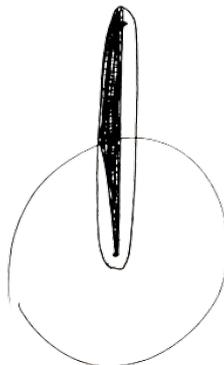
You'll usually need to apply multiple laws in a problem. Let's look at an example.

**Example 1.3** (Kevin Zhou). An object of mass  $m$  is dropped from rest at a distance  $R$  from above the Earth's surface, where  $R$  is also the radius of the Earth. Given that the mass of the Earth is  $M \gg m$ , how long does it take to hit the Earth's surface?

The solution to this uses an ingenious trick in order to utilise the idea of elliptical orbits!

Notice that the time will not change much if the mass was given a small kick in the horizontal direction before release. The orbit will hence become a very thin ellipse, with  $a \approx R$ , with one focus very near the centre of the Earth, by Kepler's First Law.

A diagram of the elliptical orbit is shown below:



The area swept out by the portion of the orbit we are interested in is the shaded area.

Let the full orbit have period  $T$ . Recall that the area of an ellipse is  $\pi ab$  where  $a$  and  $b$  are the semi-major and semi-minor axes respectively. Then, by Kepler's Second Law, using the ratio of areas, the time taken is

$$t = T \left( \frac{\frac{\pi ab}{4} + \frac{ab}{2}}{\pi ab} \right) = T \left( \frac{1}{4} + \frac{1}{2\pi} \right)$$

by summing up a quarter of an ellipse and a right triangle.

Now, the time  $T$  is given by Kepler's Third Law:

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}} = \frac{2\pi R^{\frac{3}{2}}}{\sqrt{GM}}$$

Hence, we obtain the final answer,

$$t = \left( \frac{\pi}{2} + 1 \right) \sqrt{\frac{R^3}{GM}}$$

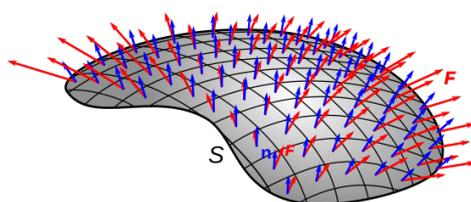
You should understand how all three of Kepler's Laws came into play in this example.

## 1.5 Ideas

Finding gravitational fields can be tricky. We shall introduce three ideas that may help.

### 1.5.1 Gravitational Flux and Gauss' Law

Loosely speaking, flux is defined as the "number of field lines passing through an area".



Mathematically speaking, the flux of a vector field  $\mathbf{F}$  is defined using a **surface integral**:

$$\Phi_F = \iint_A \mathbf{F} \cdot d\mathbf{A} \quad (35)$$

Here, note that the subscript  $A$  represents that the integral is carried out over an area. Also,  $d\mathbf{A}$  is an infinitesimal area **vector** - refer to the blue arrows in the diagram above!

When we calculate **gravitational flux**, we usually choose a surface that is **closed**, meaning that it encloses a finite, non-zero volume. Additionally, in this case, our vector field  $\mathbf{F}$  is just the gravitational field,  $\mathbf{g}$ . We hence use the following notation for gravitational flux:

$$\Phi_g = \oiint_A \mathbf{g} \cdot d\mathbf{A} \quad (36)$$

**Gauss' Law for Gravity** relates Equation (36) to the **mass enclosed** by the closed surface:

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -4\pi G m_{\text{enclosed}} \quad (37)$$

Don't be intimidated by the surface integrals! In Physics Olympiad, you'll only be applying Equation (37) to situations with **high symmetry**. These are situations where  $|\mathbf{g}|$  is constant over your chosen surface, and  $\mathbf{g}$  is purely parallel/anti-parallel to  $d\mathbf{A}$ , allowing you to simplify the dot product  $\mathbf{g} \cdot d\mathbf{A}$ .

**Remark.** Be careful of the sign conventions! For the area vector, outward is defined as positive while inward is defined as negative. Thus, outward flux is defined as positive while inward flux is defined as negative!

These surfaces with high symmetry are known as **Gaussian surfaces**. You'll need to spot **three** types of Gaussian surfaces in Olympiad problems, corresponding to different symmetries:

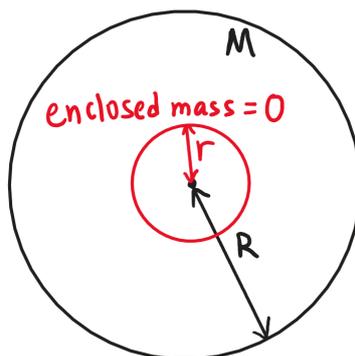
1. **Sphere:** Radially symmetric gravitational fields
2. **Cylinder:** Cylindrically symmetric gravitational fields
3. **Pillbox:** Planarly symmetric gravitational fields

**Example 1.4.** Determine the magnitude of the gravitational fields due to the following mass distributions. (i) A hollow sphere of mass  $M$  and radius  $R$ , at a point a distance  $r$  from the centre of the sphere. (ii) A solid, uniform sphere of mass  $M$  and radius  $R$ , at a point a distance  $r$  from the centre of the sphere. (iii) An infinitely long rod with uniform linear mass density  $\mu$  at a point a distance  $x$  from the rod. (iv) An infinite plane with uniform area mass density  $\sigma$  at a point a distance  $x$  from the surface.

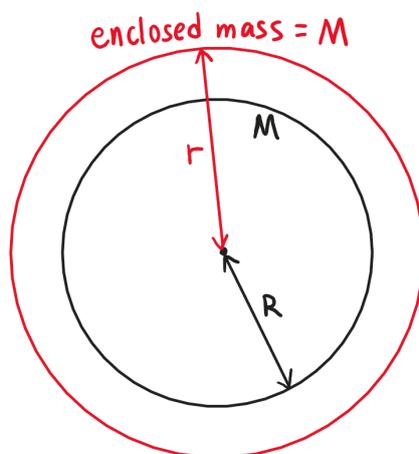
While you could have done all four parts just using naive integration (try it yourself as an exercise!), this example shows you how to apply each type of Gaussian surface to find the gravitational fields in a more elegant manner.

(i) Firstly, note that the behaviour will be different for  $r < R$  and  $r \geq R$ , as the relationship between  $m_{\text{enclosed}}$  and  $r$  differs in both cases. (Don't forget to consider both cases!) We choose our Gaussian surface to be a **concentric sphere of radius  $r$** , due to the radial symmetry.

For the former, the enclosed mass is always 0, as the Gaussian surface lies inside the hollow sphere, which has no mass inside. Thus, by Gauss' Law,  $g = 0$ .



For the latter, the enclosed mass is always  $M$ , as the Gaussian surface completely encloses the whole sphere of mass  $M$ .



Due to symmetry,  $\mathbf{g}$  always points radially inwards, and is anti-parallel with  $d\mathbf{A}$  (pointing radially outwards from the Gaussian sphere). Hence,  $\mathbf{g} \cdot d\mathbf{A} = -g dA$ . Thus, by Gauss' Law,

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -g (4\pi r^2) = -4\pi G m_{\text{enclosed}} = -4\pi GM \Rightarrow g = \frac{GM}{r^2}$$

Hence, the final answer for the gravitational field is

$$g(r) = \begin{cases} 0, & r < R \\ \frac{GM}{r^2}, & r \geq R \end{cases}$$

(ii) Again, note that the behaviour will be different for  $r < R$  and  $r \geq R$ . Again, we choose our Gaussian surface to be a **concentric sphere of radius  $r$** .

For the former, since the density is constant, the enclosed mass scales by  $r^3$ , and is  $m_{\text{enclosed}} = \frac{r^3}{R^3}M$ . Thus, by Gauss' Law,

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -g (4\pi r^2) = -4\pi G m_{\text{enclosed}} = -4\pi GM \frac{r^3}{R^3} \Rightarrow g = \frac{GMr}{R^3}$$

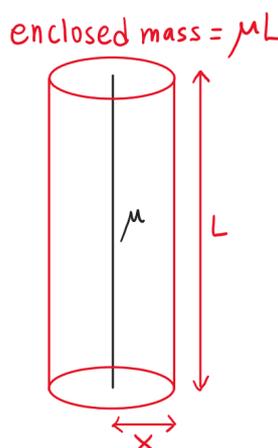
For the latter, the enclosed mass is just  $M$ . Thus, by Gauss' Law,

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -g (4\pi r^2) = -4\pi G m_{\text{enclosed}} = -4\pi GM \Rightarrow g = \frac{GM}{r^2}$$

Hence, the final answer for the gravitational field is

$$g(r) = \begin{cases} \frac{GMr}{R^3}, & r < R \\ \frac{GM}{r^2}, & r \geq R \end{cases}$$

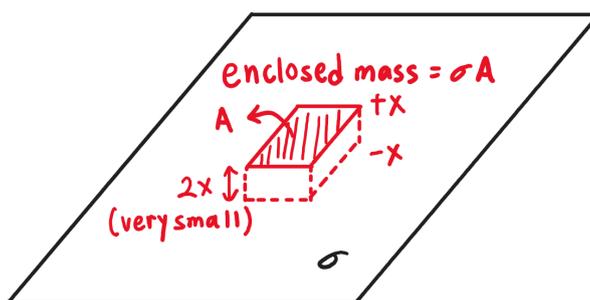
(iii) The infinite nature of the rod means that there should not be a component of the gravitational field parallel to the rod's axis. (You can think of there being an equivalent mass on the other side to cancel out any parallel component, since it is infinite.) We choose our Gaussian surface to be a **coaxial cylinder of length  $L$  and radius  $x$** , due to the cylindrical symmetry.



Due to symmetry,  $\mathbf{g}$  always points radially inwards towards the axis of the cylinders, and is anti-parallel with  $d\mathbf{A}$  (pointing radially outwards from the curved surface area of the Gaussian cylinder). Hence,  $\mathbf{g} \cdot d\mathbf{A} = -g dA$ . Thus, by Gauss' Law,

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -g(2\pi xL) = -4\pi Gm_{\text{enclosed}} = -4\pi G\mu L \quad \Rightarrow \quad g(x) = \frac{2G\mu}{x}$$

(iv) Similarly, the infinite nature of the plane means that there should not be a component of the gravitational field parallel to the plane. We choose our Gaussian surface to be a **"pillbox" (a very short box) of cross-sectional area  $A$ , with its flat faces at  $\pm x$  parallel to the plane**, due to the planar symmetry.



Due to symmetry,  $\mathbf{g}$  always points perpendicularly inwards towards the plane, and is anti-parallel with the  $d\mathbf{A}$  on the larger surfaces, and perpendicular to the  $d\mathbf{A}$  on the smaller surfaces. That is, the smaller surfaces contribute no gravitational flux, and for the larger surfaces,  $\mathbf{g} \cdot d\mathbf{A} = -g dA$ . Thus, by Gauss' Law,

$$\oiint_A \mathbf{g} \cdot d\mathbf{A} = -g(2A) = -4\pi Gm_{\text{enclosed}} = -4\pi G\sigma A \quad \Rightarrow \quad g(x) = 2\pi G\sigma$$

Surprisingly, this set-up produces a uniform gravitational field!

### 1.5.2 Newton's Shell Theorems

In Example 1.4 (i) and (ii), we have proven the following, called **Newton's Shell Theorems**:

1. **Inside** a uniform spherical shell, there is **no gravitational field** everywhere.
2. **Outside** a uniform spherical shell of total mass  $m$ , the gravitational field is the same as if the shell was replaced by a **point mass  $m$  at its centre**.

These statements are useful to simplify the calculations of gravitational fields in shells.

### 1.5.3 Superposition and Negative Mass

Sometimes, certain objects may have "holes" in them, whereby mass has been removed. Calculating the gravitational fields may be difficult, since symmetry is broken.

However, the good news is that gravitational fields obey the **principle of superposition!** This means that fields add linearly as vectors. Thus, you may calculate the gravitational fields of each piece of mass separately (using Gauss' Law or naive integration), and add them together!

The concept of **negative mass** can be applied to holes. Using the principle of superposition, we may find the gravitational field as if the hole was filled normally, and then add up the field produced by a negative mass filling up the hole.

**Example 1.5.** Consider the infinite plane in Example 1.4 (iv), except a hole of radius  $R$  is now present, centered at the origin. Find the gravitational field along the axis of the hole.

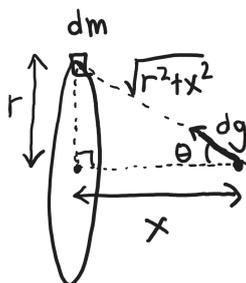
Using the principle of superposition,

$$\mathbf{g} = \mathbf{g}_{plane} + \mathbf{g}_{hole}$$

where  $\mathbf{g}_{hole}$  is produced by a hole of the same dimensions, but a negative mass density  $-\sigma$ .

We already found  $\mathbf{g}_{plane}$  in Example 1.4 (iv). To find  $\mathbf{g}_{hole}$ , let us perform naive integration, by treating the circular hole as many rings together.

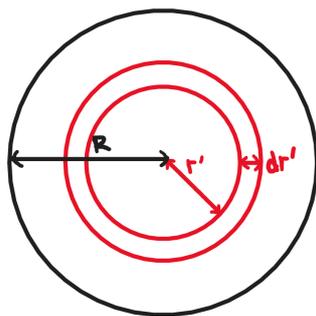
Consider a ring of radius  $r$  and mass  $m$ . We want to find the gravitational field along its axis, at a distance  $x$  away from its centre.



Clearly, after integrating across the ring, only the component  $dg \cos \theta$  will survive. From the diagram,  $\cos \theta = \frac{x}{\sqrt{r^2 + x^2}}$ . Thus,

$$g_{ring} = \int dg \cos \theta = \int \frac{G dm}{r^2 + x^2} \frac{x}{\sqrt{r^2 + x^2}} = \frac{Gx}{(r^2 + x^2)^{\frac{3}{2}}} \int dm = \frac{Gmx}{(r^2 + x^2)^{\frac{3}{2}}}$$

Now, we integrate across all the rings that form the circular hole.



The mass of each small ring is  $dm = -\sigma dA = -2\pi r' \sigma dr'$ , accounting for negative mass. Thus,

$$g_{hole} = \int \frac{Gx dm}{(r'^2 + x^2)^{\frac{3}{2}}} = - \int_0^R \frac{2\pi G\sigma x r' dr'}{(r'^2 + x^2)^{\frac{3}{2}}} = - \frac{2\pi G\sigma x}{\sqrt{x^2 + R^2}}$$

Hence, by the principle of superposition,

$$g(x) = 2\pi G\sigma - \frac{2\pi G\sigma x}{\sqrt{x^2 + R^2}} = 2\pi G\sigma \left( 1 - \frac{x}{\sqrt{x^2 + R^2}} \right)$$

Of course, you could have solved this without negative mass, by just integrating the gravitational field of the rings from  $R$  to  $\infty$ . However, this method is much more flexible, as it can apply to other situations where naive integration is difficult due to loss of symmetry!

## 1.6 References

An Introduction to Mechanics by *Kleppner & Kolenkow*

Gravitation Handout by *Kevin Zhou*

Competitive Physics by *Wang & Ricardo*

University Physics by *Young & Freedman*

An Introduction to Classical Mechanics by *David Morin*

## 2 Problems

Problems are arranged in roughly increasing difficulty. Take  $G = 6.67 \times 10^{11} \text{ Nm}^2/\text{kg}^2$  if needed.

**Problem 2.1** (SPhO 2009). An object is fired vertically upwards from the surface of the Earth (of radius  $R_E$ ) with an initial speed  $v_i$  that is comparable to but less than the escape speed  $v_{esc}$ .

(i) Show that the object attains a maximum height  $h$  given by

$$h = \frac{R_E v_i^2}{v_{esc}^2 - v_i^2}$$

(ii) A space vehicle is launched vertically upwards from the Earth's surface with an initial speed of 8.76 km/s, which is less than the escape speed of 11.2 km/s. Find the maximum height that it attains. (iii) A meteorite falls towards the Earth. It is essentially at rest with respect to the Earth when it is at a height of  $2.51 \times 10^7$  m. Determine the speed at which the meteorite strikes the Earth. (iv) Assume that a baseball is tossed up with an initial speed that is **very small** compared to the escape speed. Show that the maximum height attained by the baseball is

$$h = \frac{v_i^2}{2g}$$

where  $g$  is the acceleration due to free fall on the surface of the Earth. Show also that the equation from part (i) is consistent with this equation.

*Solution.* (i) At maximum height,  $v_f = 0$ . Thus, by COE,

$$\frac{1}{2}mv_i^2 - \frac{GMm}{R_E} = -\frac{GMm}{R_E + h} \quad \Rightarrow \quad \frac{2GM}{R_E} - v_i^2 = \frac{2GM}{R_E + h}$$

The escape speed is given by

$$\frac{1}{2}mv_{esc}^2 - \frac{GMm}{R_E} = 0 \quad \Rightarrow \quad v_{esc} = \sqrt{\frac{2GM}{R_E}}$$

Thus, rewriting in terms of  $v_{esc}$ ,

$$v_{esc}^2 - v_i^2 = \frac{2GM}{R_E + h} \quad \Rightarrow \quad h = \frac{2GM}{v_{esc}^2 - v_i^2} - R_E = \frac{2GM - R_E v_{esc}^2 + R_E v_i^2}{v_{esc}^2 - v_i^2} = \frac{R_E v_i^2}{v_{esc}^2 - v_i^2}$$

(ii) Using the result of (i), we plug in the numbers to get

$$h = 1.0 \times 10^7 \text{ m}$$

(iii) We observe that a stone thrown vertically upwards with the same speed that the meteorite strikes the Earth will also cause it to rise to the same initial height of the meteorite. (In other words, mechanics is time-reversible.) Thus,

$$v_i^2 = \frac{v_{esc}^2 h}{R_E + h} \quad \Rightarrow \quad v_i = v_{esc} \sqrt{\frac{h}{R_E + h}} = 10.0 \text{ km/s}$$

(iv) If the initial speed is very small, the maximum height is also very small, thus the gravitational field is approximately constant at  $g$  for the whole duration. This then becomes a simple 1D kinematics problem:

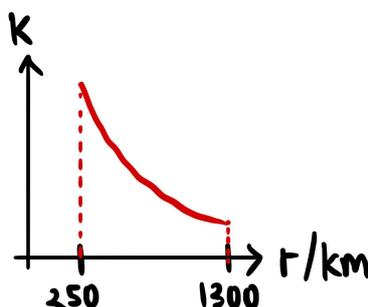
$$t = \frac{v_i}{g} \quad \Rightarrow \quad h = \frac{1}{2}gt^2 = \frac{v_i^2}{2g}$$

**Problem 2.2** (SPhO 2008). A satellite of mass 1000 kg is in an elliptical orbit around the Earth. The minimum height of the orbit above the Earth's surface is 250 km (at perigee) and the maximum height above the Earth's surface is 1300 km (at apogee). (i) Qualitatively sketch the variation of the kinetic energy of the satellite with the height above the Earth's surface from perigee to apogee. (ii) If the velocities of the satellite at apogee and at perigee are  $v_a$  and  $v_p$  respectively, using conservation of angular momentum, find  $v_a$  in terms of  $v_p$ . (iii) Using conservation of energy, hence determine the value of  $v_a$ .

*Solution.* (i) By COE,

$$\Delta K = \frac{GMm}{r_i} - \frac{GMm}{r_f}$$

Qualitatively, there is an inverse relationship between KE and distance from the Earth's surface, thus the sketch looks like:



(ii) By COAM,

$$mr_a v_a = mr_p v_p \Rightarrow r_a v_a = r_p v_p \Rightarrow v_a = \frac{r_p}{r_a} v_p = \frac{5}{26} v_p$$

(iii) By COE,

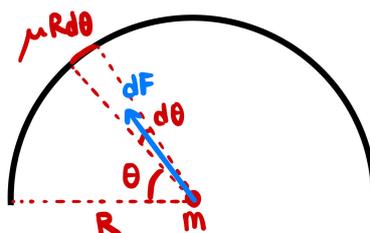
$$\frac{1}{2} m v_a^2 - \frac{GMm}{r_a} = \frac{1}{2} m v_p^2 - \frac{GMm}{r_p}$$

Substituting in the relation from (ii), you can solve to get

$$v_a = 9.93 \times 10^7 \text{ m/s}$$

**Problem 2.3.** Find the magnitude of the gravitational force acting on a point mass  $m$  located at the center of a semi-circular frame (only the arc) with linear mass density  $\mu$  and radius  $R$ .

*Solution.* It turns out the easiest way to do it is to just naively integrate the force contributions from each infinitesimal part of the semi-circular frame.



The infinitesimal force is

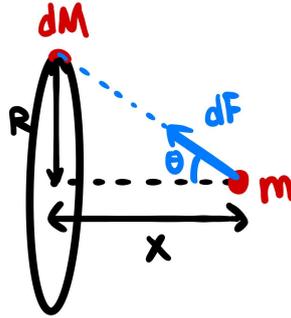
$$dF = \frac{Gm\mu R d\theta}{R} = Gm\mu d\theta$$

By symmetry, the "horizontal" component of this force cancels out upon integration, so we only care about the "vertical" component. Thus, the net force is

$$F_{net} = \int dF \sin \theta = \int_0^\pi Gm\mu \sin \theta d\theta = Gm\pi\mu$$

**Problem 2.4.** A point mass  $m$  is placed along the axis of a ring  $M$  at a distance  $x$  from the centre of the ring, and released from rest. If the mass of the ring is distributed uniformly, and the radius of the ring is  $R$ , find the angular frequency of small oscillations of the point mass along the axis of the ring, assuming  $x \ll R$ .

*Solution.* In a similar vein to the previous problem, we integrate over the ring to sum forces.



The infinitesimal force is

$$dF = \frac{Gm dM}{x^2 + R^2}$$

By symmetry, the "vertical" component of this force cancels out upon integration, so we only care about the "horizontal" component. Thus, the net force is

$$F_{net} = \int dF \cos \theta = \int \frac{Gmx dM}{(x^2 + R^2)^{\frac{3}{2}}} = \frac{Gmx}{(x^2 + R^2)^{\frac{3}{2}}} \int dM = \frac{GMmx}{(x^2 + R^2)^{\frac{3}{2}}} \approx \frac{GMmx}{R^3}$$

where the last equality invoked the limit  $x \ll R$ .

Notice that this force is restoring as it decreases  $x$ . Thus, the equation of motion is

$$m\ddot{x} + \frac{GMmx}{R^3} = 0 \quad \Rightarrow \quad \ddot{x} + \frac{GM}{R^3}x = 0$$

This is the form of a SHM, with angular frequency

$$\omega = \sqrt{\frac{GM}{R^3}}$$

**Problem 2.5** (SPhO 2019). The internal and external radii of a uniform hollow sphere are  $r$  and  $R$  respectively. Taking the gravitational potential to be zero at infinity, what is the ratio of the gravitational potential at the point on the outer surface to that on the inner surface?

*Solution.* We need to first find the expression for the gravitational field.

Let the sphere have a mass density  $\rho$ . Outside the sphere, using a Gaussian sphere gives

$$g = -\frac{GM}{x^2} = -\frac{4}{3}G\rho\pi \left( \frac{R^3 - r^3}{x^2} \right), \quad x > R$$

Inside the solid part of the sphere, using a Gaussian sphere gives

$$g = -\frac{4}{3}G\rho\pi \left( \frac{x^3 - r^3}{x^2} \right), \quad r < x < R$$

Inside the hollow part of the sphere, there is no enclosed mass, so

$$g = 0, \quad x < r$$

We can now proceed with integration. For  $x > R$ ,

$$\phi(R) = - \int_{\infty}^R \left( -\frac{4}{3}G\rho\pi \left( \frac{R^3 - r^3}{x^2} \right) \right) dx = -\frac{4}{3}G\rho\pi \left( \frac{R^3 - r^3}{R} \right)$$

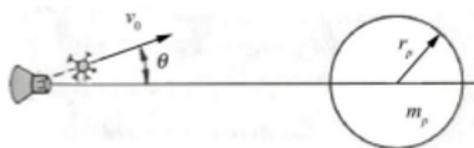
For  $r < x < R$ ,

$$\phi(r) = \phi(R) - \int_R^r \left( -\frac{4}{3}G\rho\pi \left( \frac{x^3 - r^3}{x^2} \right) \right) dx = -2G\rho\pi R^2 + 2G\rho\pi r^2$$

Thus, the desired ratio is

$$\frac{\phi(R)}{\phi(r)} = \frac{-\frac{4}{3}G\rho\pi \left( \frac{R^3 - r^3}{R} \right)}{-2G\rho\pi R^2 + 2G\rho\pi r^2} = \frac{2(R^2 + Rr + r^2)}{3R(R + r)}$$

**Problem 2.6** (Ricardo). A spaceship is sent to investigate a planet of mass  $m_p$  and radius  $r_p$ . While hanging motionless in space at a distance  $5r_p$  from the centre of the planet, the ship fires an instrument package with speed  $v_0$ . The package has mass  $m_i$  which is much smaller than the mass of the spaceship. The package is launched at an angle  $\theta$  with respect to a radial line between the centre of the planet and the spaceship. For what angle  $\theta$  will the package just graze the surface of the planet?



*Solution.* We can write the COAM equation:

$$m_i v_0 (5r_p) \sin \theta = m_i v_f r_p \quad \Rightarrow \quad v_f = 5v_0 \sin \theta$$

and we can also write the COE equation:

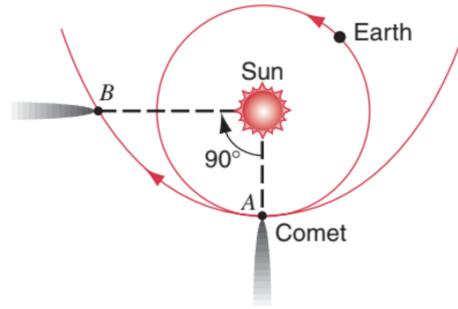
$$\frac{1}{2}m_i v_0^2 - \frac{Gm_p m_i}{5r_p} = \frac{1}{2}m_i v_f^2 - \frac{Gm_p m_i}{r_p}$$

Substituting the expression for  $v_f$  into this equation and solving, you will obtain

$$\theta = \sin^{-1} \left( \frac{1}{5} \sqrt{\frac{8Gm_p}{5r_p v_0^2} + 1} \right)$$

**Problem 2.7** (HRK). How long will it take a comet, moving in a parabolic path, to move from its point of closest approach to the Sun at  $A$ , through an angle of  $90^\circ$ , measured at the Sun, to

$B$ ? Let the distance of closest approach to the Sun be equal to the radius of the Earth's orbit, assumed circular.



*Solution.* By Kepler's First Law, the Sun is at the focus of the parabolic path. Let the comet's periapsis distance be  $R$ , the same as the Earth's orbit.

The polar equation of the parabola from the focus is

$$r(\theta) = \frac{2R}{1 + \cos \theta}$$

since a parabola has an eccentricity of 1. At  $\theta = \frac{\pi}{2}$ , the radius has swung through  $90^\circ$  and has reached  $B$ .

By Kepler's Second Law,

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

We can find this constant by evaluating it at the periapsis (which is at  $A$ ):

$$\frac{dA}{dt} = \frac{1}{2}R\sqrt{\frac{2GM}{R}} = \sqrt{\frac{GMR}{2}}$$

Using integration, we can find the area swept out:

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2}r^2 d\theta = 2R^2 \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 + \cos \theta)^2}$$

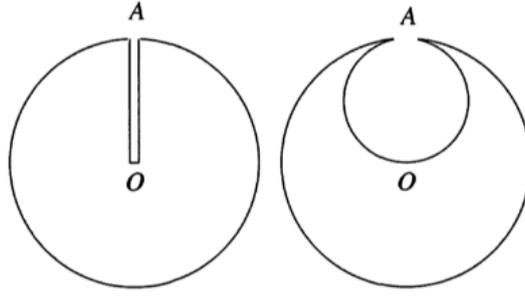
To solve this integral, we use the substitution  $u = \frac{\theta}{2}$ , thus  $\cos \theta = 1 - 2\sin^2 u$ . We get

$$A = R^2 \int_0^{\frac{\pi}{4}} \sec^4 u du = R^2 \left[ \tan u + \frac{1}{3} \tan^3 u \right]_0^{\frac{\pi}{4}} = \frac{4}{3}R^2$$

Thus, the time required is

$$t = \frac{A}{\frac{dA}{dt}} = \frac{4\sqrt{2}}{3} \frac{R^{\frac{3}{2}}}{\sqrt{GM}} = 0.30 \text{ years}$$

**Problem 2.8** (Kevin Zhou). A spaceship of titanium-devouring little green people has found a perfectly spherical homogeneous asteroid. A narrow trial shaft was bored from point  $A$  on its surface to the center  $O$  of the asteroid. At that point, one of the little green men fell off the surface of the asteroid into the trial shaft. He fell, without any braking, until he reached  $O$ , where he died on impact.



However, work continued, and the little green men started secret excavation of the titanium, in the course of which they formed a spherical cavity of diameter  $AO$  inside the asteroid. Then, a second accident occurred: another little green man similarly fell from point  $A$  to point  $O$ , and died. (i) Find the ratio of the impact speeds for the two men. (ii) Find the ratio of the times of impact for the two men.

*Solution.* (i) For the first case, we assume the volume of the hole is small and that Earth's mass and shape doesn't change from it.

At a radius  $r$  from the centre, by Gauss' Law,

$$g = -\frac{GMr}{R^3} \Rightarrow F = mg = -\frac{GMm}{R^3}r = m\ddot{r} \Rightarrow \ddot{r} + \frac{GM}{R^3}r = 0$$

This is the form of a SHM with

$$\omega = \sqrt{\frac{GM}{R^3}} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^3}{GM}}$$

The time taken to fall is a quarter of the period:

$$t_1 = \frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{R^3}{GM}}$$

while the impact speed is

$$v_1 = R\omega = \sqrt{\frac{GM}{R}}$$

For the second case, we apply superposition, treating the hole as a negative mass  $-\frac{1}{8}M$ . When the object is at a radius  $r$  from the centre of the big sphere, it is at a radius  $r - \frac{R}{2}$  from the centre of the hole. Hence,

$$g = -\frac{GMr}{R^3} + \frac{G\left(\frac{1}{8}M\right)\left(r - \frac{R}{2}\right)}{\left(\frac{R}{2}\right)^3} = -\frac{GM}{2R^2}$$

This is a case of constant acceleration, so the time taken is

$$t_2 = \sqrt{\frac{2R}{g}} = \frac{2R^{\frac{3}{2}}}{\sqrt{GM}}$$

while the impact speed is

$$v_2 = gt = \sqrt{\frac{GM}{R}}$$

The desired ratio of impact speeds is

$$\frac{v_1}{v_2} = 1$$

(ii) The desired ratio of impact times is

$$\frac{t_1}{t_2} = \frac{\pi}{4}$$

**Problem 2.9** (SPhO 2017). A planet is in a circular orbit around a star of mass  $M$ . The star is much more massive than the planet. The star explodes symmetrically and instantaneously such that its outer envelope is pushed far beyond the orbit of the planet (and does not interfere with the planet). The remaining piece of the star is of mass  $M'$  (still much greater than the mass of the planet). Find the eccentricity of the new orbit of the planet.

*Solution.* Let  $R$  be the initial radius of the orbit and  $m$  be the mass of the orbiting object. Then, after the change in mass from  $M$  to  $M'$ , the periapsis is  $r_p = R$  while the apoapsis is  $r_a > r_p$ .

The initial orbital speed of the mass is

$$v = \sqrt{\frac{GM}{R}}$$

Note that this speed **cannot instantaneously change** after the explosion! (That will imply an infinite acceleration, which is not physical.)

Hence, we can write the COE equation for the elliptical orbit:

$$\frac{1}{2}mv^2 - \frac{GmM'}{R} = \frac{1}{2}mv_a^2 - \frac{GmM'}{R_a} \Rightarrow v^2 - \frac{2GM'}{R} = v_a^2 - \frac{2GM'}{R_a}$$

and we can also write the COAM equation:

$$mRv = mR_av_a \Rightarrow Rv = R_av_a$$

Putting both equations together,

$$Rv = R_a \sqrt{v^2 - 2GM' \left( \frac{1}{R} - \frac{1}{R_a} \right)}$$

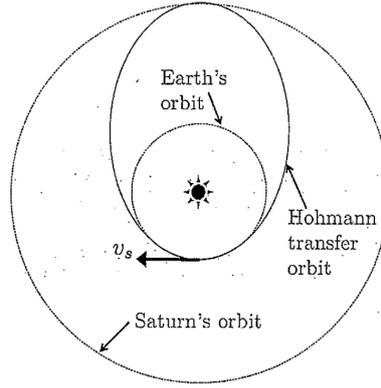
This can be rearranged into a quadratic equation:

$$\left(1 - \frac{2M'}{M}\right) \left(\frac{R_a}{R}\right)^2 + \frac{2M'}{M} \left(\frac{R_a}{R}\right) - 1 = 0 \Rightarrow R_a = R \text{ or } R_a = \frac{R}{\frac{2M'}{M} - 1}$$

The eccentricity is hence given by

$$\varepsilon = \frac{R_a - R}{R_a + R} = \frac{M'}{M} - 1$$

**Problem 2.10** (SPhO 2016). A spacecraft is to be sent from Earth to Saturn. The Hohmann transfer is a manoeuvre that transfers the spacecraft from Earth's orbit around the Sun (assumed to be circular) to Saturn's orbit around the Sun (assumed to be circular also). This is done by pushing the spacecraft into an elliptical orbit that has its perihelion and aphelion touching the Earth's orbit and Saturn's orbit, respectively.



Through this problem, you may use the following symbols and constants in your working.

1. Earth's velocity around the Sun:  $v_{\oplus} = 2.97 \times 10^4 \text{ m/s}$
2. Earth's orbital radius around the Sun:  $r_{\oplus} = 1.50 \times 10^{11} \text{ m}$
3. Saturn's orbital radius around the Sun:  $r_{\text{h}} = 1.43 \times 10^{12} \text{ m}$
4. Mass of the Sun:  $M_{\odot} = 1.99 \times 10^{30} \text{ kg}$
5. Venus' orbital radius around the Sun:  $r_{\text{v}} = 1.08 \times 10^{11} \text{ m}$
6. Venus' orbital period around the Sun:  $T_{\text{v}} = 0.615 \text{ years}$

(i) Assuming that the spacecraft is initially travelling at the velocity of the Earth around the Sun, calculate the change in velocity  $\Delta v$  for the spacecraft to transfer from Earth's orbit to the elliptical orbit. Ignore the Earth's and Saturn's gravitational forces. (ii) Calculate the time it takes for the spacecraft to travel from Earth's orbit to Saturn's orbit. (iii) The Cassini spacecraft was not sent to Saturn by a direct Hohmann transfer. Instead, it was first sent to Venus (via Hohmann transfer), where it gained speed twice, using gravity assists by Venus. The first gravity assist by Venus changed the spacecraft's velocity and transferred the spacecraft to a more eccentric orbit, such that the spacecraft returned to its orbital perihelion after Venus had completed its orbit twice, and the spacecraft received the second gravity assist by Venus at that point. Calculate the change in velocity of the spacecraft during the first Venus' gravity assist.

*Solution.* (i) The final semi-major axis is  $a = \frac{r_{\oplus} + r_{\text{h}}}{2}$ . Thus, by COE,

$$\frac{1}{2}m(v_{\oplus} + \Delta v)^2 - \frac{1}{2}mv_{\oplus}^2 = \frac{GM_{\odot}m}{2r_{\oplus}} - \frac{GM_{\odot}m}{r_{\oplus} + r_{\text{h}}}$$

Thus, the change in velocity is

$$\Delta v = \sqrt{v_{\oplus}^2 + GM_{\odot} \left( \frac{1}{r_{\oplus}} - \frac{2}{r_{\oplus} + r_{\text{h}}} \right)} - v_{\oplus} = 1.03 \times 10^4 \text{ m/s}$$

(ii) The time taken for the Hohmann transfer orbit is half the period. By Kepler's Third Law,

$$t = \frac{T}{2} = \frac{\pi a^{\frac{3}{2}}}{\sqrt{GM_{\odot}}} = \frac{\pi \left( \frac{r_{\oplus} + r_{\text{h}}}{2} \right)^{\frac{3}{2}}}{\sqrt{GM_{\odot}}} = 6.07 \text{ years}$$

(iii) The initial semi-major axis is  $\frac{r_{\oplus} + r_{\text{v}}}{2}$ . By Kepler's Third Law,  $T \propto a^{\frac{3}{2}}$ , thus the final semi-major axis is  $2^{\frac{2}{3}}r_{\text{v}}$ . The new velocity when sent to Venus via Hohmann transfer is

$$v_1 = \sqrt{GM_{\odot} \left( \frac{2}{r_{\oplus}} - \frac{2}{r_{\oplus} + r_{\text{v}}} \right)}$$

Upon intersecting Venus' orbit for the first assist, by COAM, its velocity is

$$v_2 = \frac{r_{\oplus}}{r_{\ominus}} v_1$$

The speed needed to transit to the new orbit with semi-major axis  $a = 2^{\frac{2}{3}} r_{\oplus}$  is

$$v_3 = \sqrt{GM_{\odot} \left( \frac{2}{r_{\oplus}} - \frac{1}{2^{\frac{2}{3}} r_{\oplus}} \right)}$$

The change in velocity is

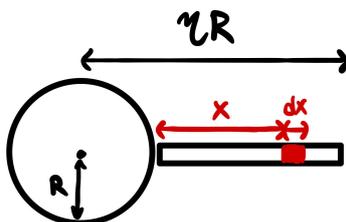
$$\Delta v = v_3 - v_2 = 3230 \text{ m/s}$$

**Problem 2.11** (Morin). Let the Earth's radius be  $R$ , its average density be  $\rho$ , and its angular frequency of rotation be  $\omega$ . Consider a long rope with uniform mass density extending radially from just above the surface of the Earth out to a radius of  $\eta R$ . (i) Show that if the rope is to remain above the same point on the Equator at all times, then we must have

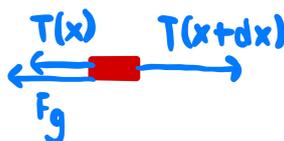
$$\eta^2 + \eta = \frac{8\pi G\rho}{3\omega^2}$$

(ii) Where does the tension along the rope achieve its maximum value?

*Solution.* (i) For the rope to remain synchronous with the Earth's rotation,  $\omega_{\text{rope}} = \omega_{\text{Earth}} = \omega$ . Let's analyse forces on infinitesimal portions of the rope.



The FBD of the infinitesimal segment  $dx$  is as follows:



Accounting for the net centripetal acceleration towards the centre of the Earth,

$$T(x+dx) - T(x) = \frac{GM\rho_r}{x^2} dx - \rho_r \omega^2 x dx \quad \Rightarrow \quad \frac{dT}{dx} = \frac{GM\rho_r}{x^2} - \rho_r \omega^2 x$$

We can solve this ODE by separating variables:

$$\int dT = \int \left( \frac{GM\rho_r}{x^2} - \rho_r \omega^2 x \right) dx \quad \Rightarrow \quad T(x) = -\frac{GM\rho_r}{x} - \frac{\rho_r \omega^2 x^2}{2} + C$$

The constant  $C$  can be found using the boundary conditions. At the ends of the rope, since they are connected to nothing, tension must be 0. Thus,  $T(R) = 0$  and  $T(\eta R) = 0$ .

Substituting these conditions in,

$$-\frac{GM\rho_r}{R} - \frac{\rho_r\omega^2 R^2}{2} + C = 0 \Rightarrow C = \frac{GM\rho_r}{R} + \frac{\rho_r\omega^2 R^2}{2}$$

$$-\frac{GM\rho_r}{\eta R} - \frac{\rho_r\omega^2\eta^2 R^2}{2} + \frac{GM\rho_r}{R} + \frac{\rho_r\omega^2 R^2}{2} = 0 \Rightarrow \frac{GM\rho_r}{R} \left(1 - \frac{1}{\eta}\right) = \frac{\rho_r\omega^2 R^2}{2} (\eta^2 - 1)$$

$$\Rightarrow \frac{G}{R} \left(\frac{4}{3}\pi R^3 \rho\right) \left(1 - \frac{1}{\eta}\right) = \frac{\omega^2 R^2}{2} (\eta^2 - 1) \Rightarrow \eta^2 + \eta = \frac{8G\pi\rho}{3\omega^2} \text{ (proven)}$$

By substituting in numerical values, you can find  $\eta \approx 23.6$ .

(ii) The tension achieves its maximum value when  $\frac{dT}{dx} = 0$ . Thus,

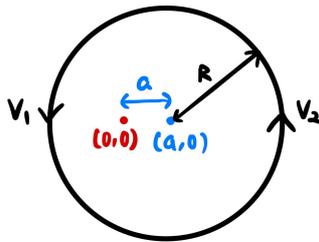
$$\frac{GM}{x^2} = \omega^2 x \Rightarrow x = \sqrt[3]{\frac{GM}{\omega^2}} = R\sqrt[3]{\frac{4G\rho\pi}{3\omega^2}}$$

To be rigorous, you should also check that this stationary point is a local maximum! (It is.)

**Problem 2.12** (Ricardo). The trajectory of a particle under the influence of a central force (not necessarily gravitational) is a circle of radius  $R$ . The position of the source of the central force is not known. Given that the maximum and minimum speeds of the particle are  $v_1$  and  $v_2$  respectively, determine the period  $T$  of the particle's orbit. (Hint: Kepler's 2nd Law holds for *all* central forces.)

*Solution.* It may seem puzzling at first how the speed of the particle varies even along a circular orbit. This is because the source of the central force is *not* at the centre!

We define our coordinate system with the source at the origin and the centre of the circle at  $(a, 0)$ , where  $a \neq 0$ , as such:



By conserving angular momentum per unit mass about the source,

$$v_1 (R - a) = v_2 (R + a) = h$$

As per the hint, we apply Kepler's Second Law, so the rate of area swept out by the particle is

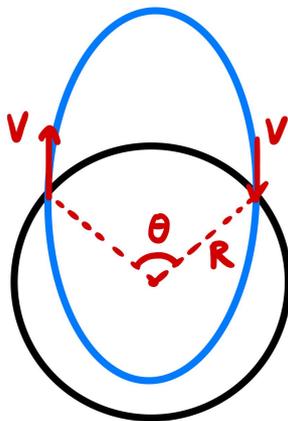
$$\frac{dA}{dt} = \frac{h}{2} = \frac{Rv_1v_2}{v_1 + v_2}$$

Separating variables and then integrating over one period  $T$ ,

$$\int_0^{\pi R^2} dA = \int_0^T \frac{Rv_1v_2}{v_1 + v_2} dt \Rightarrow T = \frac{\pi R (v_1 + v_2)}{v_1 v_2}$$

**Problem 2.13** (200 Puzzling Physics Problems). A rocket is launched from and returns to a spherical planet of radius  $R$  so that its velocity vector on return is parallel to its velocity vector at launch. The angular separation at the center of the planet between the launch and arrival points is  $\theta$ . How long does the flight take, if the period of a satellite flying around the planet just above its surface is  $T_0$ ?

*Solution.* We first draw a diagram of the trajectory of the rocket:



By Kepler's First Law, the starting and ending points of the trajectory are the two ends of the minor axis, and the centre of the planet is one of the foci of the ellipse.

The distance from the focus to the ends of the minor axis is  $R$ . Thus, the sum of distances from the two foci (recall there is one empty focus) to one end of the minor axis is  $2R$ .

Hence, we can relate the semi-major axis:

$$2a = 2R \quad \Rightarrow \quad a = R$$

We now invoke Kepler's Second Law. The area swept out by the trajectory is the sum of the area of half an ellipse and a triangle:

$$A = \frac{1}{2}A_{\text{ellipse}} + \frac{1}{2}a^2 \sin \theta = \frac{1}{2}\pi a^2 \sin^2 \frac{\theta}{2} + \frac{1}{2}a^2 \sin^2 \theta$$

The full period of this elliptical orbit is equal to  $T_0$  because the circular orbit's radius is also  $R$ . Thus, the time taken for this flight is given by the ratio of areas swept out:

$$T = T_0 \left( \frac{\frac{1}{2}\pi a^2 \sin^2 \frac{\theta}{2} + \frac{1}{2}a^2 \sin^2 \theta}{\pi a^2 \sin^2 \frac{\theta}{2}} \right) = T_0 \left( \frac{1}{2} + \frac{\cos \frac{\theta}{2}}{\pi} \right)$$

### 3 Advanced Problems

These problems are way too difficult to be tested in a modern-day SPhO. If you have completed all the previous problems and are down for a challenge, try these!

**Problem 3.1** (Kevin Zhou). We shall investigate the [first](#), [second](#) and [third cosmic velocities](#). Evaluate all your answers numerically, using the following values:

1. Mass of the Earth:  $M_{Earth} = 5.97 \times 10^{24}$  kg
2. Mass of the Sun:  $M_{Sun} = 1.99 \times 10^{30}$  kg
3. Radius of the Earth:  $R_{Earth} = 6.37 \times 10^6$  m
4. Distance between the Sun and the Earth:  $d_{Sun} = 1.50 \times 10^{11}$  m

(i) Starting from the Earth's surface, what is the minimum launch speed required to put a satellite into orbit around Earth? This is the first cosmic speed. Do not account for the rotation of the Earth. (ii) Now, accounting for the rotation of the Earth, what is this new minimum speed, and how should the satellite be launched? (iii) What is the minimum launch speed required for a rocket to escape the gravitational field of the Earth? This is the second cosmic speed. (iv) What is the minimum launch speed required for a rocket to leave the solar system? This is the third cosmic speed. How should the satellite be launched? Work in the approximation that  $R_{Earth} \ll d_{Sun}$ . (v) What is the minimum launch speed for a rocket to hit the Sun? Assume that you cannot make any adjustments to the rocket's path after launch. (vi) If subsequent adjustments are allowed, the minimum launch speed in (v) can be dramatically reduced. Find the new minimum launch speed if an infinitesimal adjustment later is allowed.

**Remark.** You should only be finding (iv)-(vi) difficult.

*Solution.* (i) This is trivially

$$v_1 = \sqrt{\frac{GM_{Earth}}{R_{Earth}}} = 7.9 \text{ km/s}$$

(ii) The Earth imparts a speed of  $v_r$  from its rotation. The most optimal is launching from the Equator (latitude  $90^\circ$ ), as we only need to launch with speed  $v_1 - v_r$ .

We have

$$v_r = R_{Earth}\omega_r = \frac{2\pi R_{Earth}}{T_r} = 0.46 \text{ km/s}$$

and thus,

$$v_{1,min} = v_1 - v_r = 7.4 \text{ m/s}$$

(iii) This is also trivially

$$v_2 = \sqrt{\frac{2GM_{Earth}}{R_{Earth}}} = 11.2 \text{ km/s}$$

(iv) There is a very easy way to obtain the **wrong solution**, illustrated below.

**Wrong solution:** To escape the solar system, the rocket must escape the Sun too. As Earth moves at  $v_{Earth} = \sqrt{\frac{GM_{Sun}}{d_{Sun}}}$ , the fastest part of the Earth moves at  $v_{Earth} + R_{Earth}\omega_r$ .

Hence, COE gives

$$\frac{1}{2}m(v_{Earth} + R_{Earth}\omega_r + v_3)^2 - \frac{GM_{Earth}m}{R_{Earth}} - \frac{GM_{Sun}m}{d_{Sun}} = 0$$

$$\Rightarrow v_3 = \sqrt{2G \left( \frac{M_{Earth}}{R_{Earth}} + \frac{M_{Sun}}{d_{Sun}} \right)} - v_{Earth} - R_{Earth}\omega_r$$

The reason why this is wrong is because COE cannot be applied in *one* step! We would necessarily have to work in a frame whereby either the Earth or the Sun has a significant velocity if we did this, leading to large changes in the KE of the Earth or the Sun, which are messy to deal with.

The correct way is to work in *two* stages: first, the rocket leaves the field of the Earth, then it leaves the field of the Sun.

**Correct solution:** Once the rocket leaves the field of the Earth, its velocity relative to the Sun must be  $\sqrt{\frac{2GM_{Sun}}{d_{Sun}}}$  to escape the Sun.

Since the Earth already has velocity  $\sqrt{\frac{GM_{Sun}}{d_{Sun}}}$ , the minimum relative velocity to Earth is  $(\sqrt{2} - 1) \sqrt{\frac{GM_{Sun}}{d_{Sun}}}$ .

Now, work in the frame of the Earth in the first stage. If the launch velocity is  $v_3$ , COE gives

$$\frac{1}{2}m \left( v_3^2 - \left( (\sqrt{2} - 1) \sqrt{\frac{GM_{Sun}}{d_{Sun}}} \right)^2 \right) = \frac{GM_{Earth}m}{R_{Earth}} \Rightarrow v_3 = 16.7 \text{ km/s}$$

If you had done the wrong solution (or something similar), don't worry! The author of this handout also did the wrong solution when he was learning this. Reference frames and energy are tricky indeed :p

(v) In this case, after leaving Earth, we need zero velocity. This means we need a velocity of  $\sqrt{\frac{GM_{Sun}}{d_{Sun}}}$ . Thus, by a similar reasoning,

$$v = \sqrt{\left( \sqrt{\frac{2GM_{Earth}}{R_{Earth}}} \right)^2 + \left( \sqrt{\frac{GM_{Sun}}{d_{Sun}}} \right)^2} = 31.8 \text{ km/s}$$

(vi) The best option is to perform the operation described in (iv), to first escape the solar system. Afterwards, when the rocket is far away, it can perform an infinitesimal boost to cancel out its speed, and it will then fall into the Sun.

**Problem 3.2** (Physics Cup). A cannonball is launched from the Equator and subsequently hits the North Pole. Neglecting air resistance and the Earth's rotation, at what angle to the horizontal should the cannonball be fired to minimise the required speed? (The North Pole and the Equator differ by a latitude of  $90^\circ$ .)

*Solution.* You may find the official solutions [here](#).

**Problem 3.3** (APhO 2017 T2). An excellent gravitation problem. This problem is representative of the level of international Olympiads, so don't be too discouraged if you can't solve parts of it!

*Solution.* You may find the official solutions [here](#).

## 4 Appendix

### 4.1 Deriving the Total Energy of an Elliptical Orbit

Let us introduce the quantities below, which will be useful for the proof:

1. **Specific Angular Momentum:**  $J = \frac{L}{m}$ , defined as the angular momentum per unit mass.
2. **Specific Energy:**  $\epsilon = \frac{E}{m}$ , defined as the (total) energy per unit mass.

Clearly, since  $L$  and  $E$  are conserved, thus  $J$  and  $\epsilon$  are conserved too.

Let the periapsis distance be  $r_1$  and the apoapsis distance be  $r_2$ . By conserving  $J$ ,

$$J = r_1 v_1 = r_2 v_2 \quad (38)$$

and by conserving  $2\epsilon$  at these two points,

$$2\epsilon = v_1^2 - \frac{2GM}{r_1} = v_2^2 - \frac{2GM}{r_2} \quad (39)$$

Thus, combining Equations (38) and (39),

$$\epsilon = \frac{J^2}{2r^2} - \frac{GM}{r} \quad \Rightarrow \quad \epsilon r^2 + GMr - \frac{J^2}{2} = 0 \quad (40)$$

whereby Equation (40) is satisfied by both  $r = r_1$  and  $r = r_2$ .

By Vieta's formula,

$$\text{Sum of roots} = r_1 + r_2 = -\frac{GM}{\epsilon} = -\frac{GMm}{E} \quad (41)$$

Recall that  $r_1 + r_2 = 2a$ . Thus,

$$2a = -\frac{GMm}{E} \quad \Rightarrow \quad E = -\frac{GMm}{2a} \quad (42)$$

as desired. This concludes the proof.

### 4.2 A Discussion on Central Forces

In general, a **central force** is defined as any force that points **purely radially** towards or away from a fixed point in space. For instance, clearly, the gravitational force is a central force.

Since we care about the radial direction, let's turn our focus to polar coordinates. Writing N2L in polar coordinates,

$$F_r = m(\ddot{r} - r\dot{\theta}^2) \quad (43)$$

$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \quad (44)$$

Since we are dealing with a central force,  $F_\theta = 0$  and  $F_r = F(r)$ . Equation (44) becomes

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad \Rightarrow \quad r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \quad \Rightarrow \quad \frac{d}{dt}(r^2\dot{\theta}) = 0 \quad \Rightarrow \quad r^2\dot{\theta} = \text{constant} \quad (45)$$

Recall that  $L = mr^2\dot{\theta}$ . Thus, this implies that  $L$  is conserved, as we expect.

With  $L$ , we may analyse the total energy of a system under the influence of a central force. The total energy is given by

$$E = K + U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) = \frac{1}{2}m\dot{r}^2 + \left( U(r) + \frac{L^2}{2mr^2} \right) \quad (46)$$

Notice how we grouped the terms together. The term in the brackets can be defined as the **effective potential energy**,  $U_{eff}(r)$ :

$$U_{eff}(r) := U(r) + \frac{L^2}{2mr^2} \quad (47)$$

This is convenient because the effective potential energy only depends on  $r$ , while the other term,  $\frac{1}{2}m\dot{r}^2$ , only depends on  $\dot{r}$ . Hence, the term outside the brackets more closely resembles kinetic energy, while the term inside the brackets more closely resembles potential energy!

**Remark.** The  $\frac{L^2}{2mr^2}$  term is actually kinetic energy, but it is more convenient to just treat it as part of the potential energy.

We can also note that  $U_{eff}(r) = E - \frac{1}{2}m\dot{r}^2$ . As  $\frac{1}{2}m\dot{r}^2 \geq 0$ , thus

$$U_{eff}(r) \leq E \quad (48)$$

hence, setting an upper bound for  $U_{eff}(r)$ .

This idea is useful for finding apsides. At the apsides, the radial distance is maximal/minimal, and hence  $\dot{r} = 0$ . Thus,  $E = U_{eff}$ .

Going back to forces, we can use the substitution  $\dot{\theta} = \frac{L}{mr^2}$ , thus Equation (43) becomes

$$F(r) = m \left( \ddot{r} - r \left( \frac{L}{mr^2} \right)^2 \right) = m \left( \ddot{r} - \frac{L^2}{m^2 r^3} \right) \quad (49)$$

which is a differential equation solely in  $r$ . We have reduced our problem in  $r$  and  $\theta$  to just  $r$ !

However, this is still not easy to solve. (You may try it as an exercise if you aren't convinced how hard it is to solve this.) Thus, let's turn our focus to an ingenious substitution, called the **inverse radial position**,  $u$ :

$$u := \frac{1}{r} \quad (50)$$

Since  $dr = -\frac{1}{u^2} du$ , we may convert the position-derivative operator:

$$\frac{d}{dr} = -u^2 \frac{d}{du} \quad (51)$$

We also know from COAM that

$$L = mr^2 \frac{d\theta}{dt} \Rightarrow \frac{d}{dt} = \frac{L}{mr^2} \frac{d}{d\theta} = \frac{Lu^2}{m} \frac{d}{d\theta} \quad (52)$$

**Remark.** At this point, it is worth noting that Equations (51) and (52) alone make no mathematical sense. Instead, here, we are dealing with **operators** - the  $\frac{d}{dr}$  and  $\frac{d}{dt}$  are operators which will be applied to quantities.

Now, applying Equation (52), we differentiate  $r$  with respect to time:

$$\dot{r} = \frac{Lu^2}{m} \frac{d}{d\theta} \left( \frac{1}{u} \right) = -\frac{Lu^2}{m} \left( -\frac{1}{u^2} \frac{du}{d\theta} \right) = -\frac{L}{m} \frac{du}{d\theta} \quad (53)$$

Differentiating with respect to time once more,

$$\ddot{r} = \frac{Lu^2}{m} \frac{d}{d\theta} \left( -\frac{L}{m} \frac{du}{d\theta} \right) = -\frac{L^2 u^2}{m^2} \frac{d^2 u}{d\theta^2} \quad (54)$$

Thus, Equation (49) reduces to

$$F(u) = -\frac{L^2 u^2}{m} \frac{d^2 u}{d\theta^2} - \frac{L^2 u^3}{m} \Rightarrow \frac{d^2 u}{d\theta^2} + u = -\frac{m}{L^2 u^2} F(u) \quad (55)$$

or, equivalently, using the idea of effective potential energy in Equation (47),

$$\frac{d^2 u}{d\theta^2} = -\frac{m}{L^2} U'_{eff}(u) \quad (56)$$

This is a 2nd order ODE that we are more familiar with, and is commonly referred to as the **Binet Equation**.

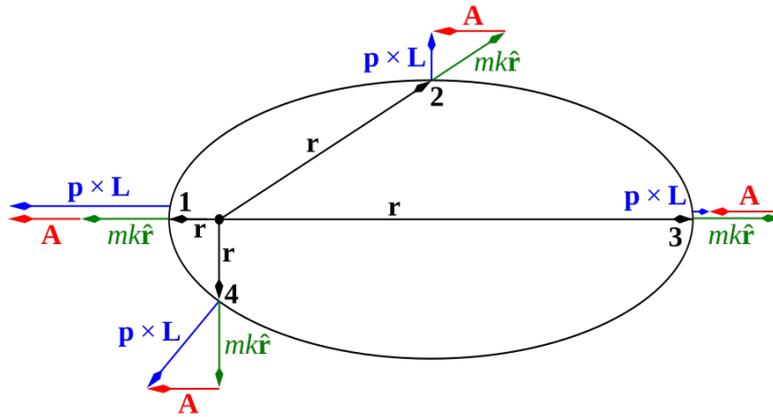
What remains is to substitute in  $F(u)$  and solve for  $u(\theta)$ , and hence  $r(\theta)$ . You have solved for the polar equation of the trajectory of a particle under a central force!

### 4.3 Laplace-Runge-Lenz Vector

The **Laplace-Runge-Lenz vector**

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - GMm^2 \hat{\mathbf{r}} \quad (57)$$

is a special quantity that is **conserved**, given a star of mass  $M$  at the origin, and a planet with radial position  $\mathbf{r}$ , momentum  $\mathbf{p}$  and angular momentum  $\mathbf{L}$ .



To show that this is conserved, we consider the time-derivative of  $\mathbf{A}$ . Any conserved quantity will have its time-derivative equal to 0, so our goal is to prove that it is 0.

Recall that

$$\mathbf{p} = m\mathbf{v}, \quad \mathbf{L} = m\mathbf{r} \times \mathbf{v}, \quad \frac{d\mathbf{p}}{dt} = \mathbf{F}$$

Thus, we can evaluate

$$\mathbf{p} \times \mathbf{L} = m\mathbf{v} \times (m\mathbf{r} \times \mathbf{v}) = m^2 \mathbf{v} \times (\mathbf{r} \times \mathbf{v}) = m^2 (v^2 \mathbf{r} - \mathbf{v}(\mathbf{r} \cdot \mathbf{v})) \quad (58)$$

where we have used the [vector triple product](#) for the last equality.

Now, we can differentiate

$$\frac{d\mathbf{A}}{dt} = \frac{d}{dt} (\mathbf{p} \times \mathbf{L} - GMm^2\hat{\mathbf{r}}) = \frac{d}{dt} (m^2 (v^2\mathbf{r} - \mathbf{v}(\mathbf{r} \cdot \mathbf{v})) - GMm^2\hat{\mathbf{r}}) \quad (59)$$

We shall simplify term by term:

$$\frac{d}{dt} (v^2\mathbf{r}) = \frac{d}{dt} ((\mathbf{v} \cdot \mathbf{v})\mathbf{r}) = (2\mathbf{v} \cdot \mathbf{a})\mathbf{r} + v^2\mathbf{v} \quad (60)$$

$$\frac{d}{dt} ((\mathbf{r} \cdot \mathbf{v})\mathbf{v}) = (v^2 + \mathbf{r} \cdot \mathbf{a})\mathbf{v} + (\mathbf{r} \cdot \mathbf{v})\mathbf{a} \quad (61)$$

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d}{dt} \left( \frac{\mathbf{r}}{r} \right) = \frac{\mathbf{v}}{r} - \frac{\mathbf{r}}{r^2} \frac{dr}{dt} = \frac{\mathbf{v}}{r} - \frac{\mathbf{r}}{r^2} (\hat{\mathbf{r}} \cdot \mathbf{v}) = \frac{\mathbf{v}}{r} - \frac{(\mathbf{r} \cdot \mathbf{v})\mathbf{r}}{r^3} \quad (62)$$

Note that in Equation (62), the second equality is true from the definition of radial speed:

$$\frac{dr}{dt} = \frac{d}{dt} (\sqrt{\mathbf{r} \cdot \mathbf{r}}) = \frac{1}{2r} \frac{d(\mathbf{r} \cdot \mathbf{r})}{dt} = \frac{\mathbf{r} \cdot \mathbf{v}}{r} = \hat{\mathbf{r}} \cdot \mathbf{v}$$

Now, recall that

$$\mathbf{a} = \ddot{\mathbf{r}} = \frac{\mathbf{F}}{m} = -\frac{GM}{r^2}\hat{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r} \quad (63)$$

Thus, the full simplification of Equation (59) is as such:

$$\begin{aligned} \frac{d\mathbf{A}}{dt} &= m^2 (2(\mathbf{v} \cdot \mathbf{a})\mathbf{r} - (\mathbf{r} \cdot \mathbf{a})\mathbf{v} - (\mathbf{r} \cdot \mathbf{v})\mathbf{a}) - GMm^2\frac{\mathbf{v}}{r} + GMm^2\frac{\mathbf{r} \cdot \mathbf{v}}{r^3}\mathbf{r} \\ &= m^2 \left( -\frac{2GM}{r^3}(\mathbf{r} \cdot \mathbf{v})\mathbf{r} + \frac{GM}{r}\mathbf{v} + \frac{GM}{r^3}(\mathbf{r} \cdot \mathbf{v})\mathbf{r} \right) - GMm^2\frac{\mathbf{v}}{r} + GMm^2\frac{\mathbf{r} \cdot \mathbf{v}}{r^3}\mathbf{r} \\ &= m^2 \left( -\frac{GM}{r^3}(\mathbf{r} \cdot \mathbf{v})\mathbf{r} + \frac{GM}{r}\mathbf{v} \right) - GMm^2\frac{\mathbf{v}}{r} + GMm^2\frac{\mathbf{r} \cdot \mathbf{v}}{r^3}\mathbf{r} = 0 \end{aligned}$$

which is exactly as desired, thus proving that  $\mathbf{A}$  is a constant of motion.

This may seem extremely contrived, but the Laplace-Runge-Lenz vector can be elegantly used to solve some problems. For a very hard problem that uses this idea, you may look at [Physics Cup 2021, Problem 2](#).